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1992 J. Phys. A: Math. Gen. 25 L1181

(http://iopscience.iop.org/0305-4470/25/19/009)

J. Phys. A: Math. Gen. 25 (1992) L1181–L1185. Printed in the UK

LETTER TO THE EDITOR

Exact results for fully directed polymer networks in two dimensions

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Received 5 May 1992

Abstract. We study a connected polymer network in two dimensions with a specified topology consisting of identical long and fully directed chains. The exact values of the bulk critical exponent, γ_G and the surface critical exponent γ'_G are obtained rigorously.

Polymer networks made from long chains and subject to the self-avoiding constraint, have been studied in bulk and in a semi-infinite good solvent (see De'Bell and Lookman 1992 for a review). It has been shown (Saleur 1986, Duplantier 1986, Duplantier and Saleur 1986, and Ohno and Binder 1988) that the number of configurations $W_N(t)$ of a network G in which all N chains have the same length t, has the asymptotic form

$$W_N(t) \approx C\mu^{Nt}t^{\gamma_G - 1} \tag{1}$$

where the critical exponent γ_G is a sum of independent contributions from each L-leg vertex expressed in terms of a scaling dimension Δ_L which depends on L but not on G. For the semi-infinite system the value of μ is the same as for the bulk and the corresponding exponent γ'_G may be decomposed in a similar way but vertices attached to the surface have a different scaling dimension Δ'_L.

Here we consider connected networks of fully directed chains in which each link has a positive component parallel to some chosen direction. In the semi-infinite system this direction is parallel to the surface. At an L-leg vertex of a such a network the L^- chains flowing into the vertex and the L^+ chains emanating from it are totally independent of each other (except for their common vertex). In other words, we can decompose such an L-leg vertex into an incoming L^--leg fan and an outgoing L^+-leg fan with $L^- + L^+ = L$ and it would therefore be expected that each part would make its own separate contribution to the critical exponents.

If we let $\bar{n}_L(\bar{n}'_L)$ be the total number of both incoming and outgoing L-leg fans in the bulk (surface), then we show in this work that for such a network in two dimensions the critical exponents γ_G and γ'_G are given by

$$\gamma_G - 1 = -\tfrac{1}{2}\mathcal{L} - \tfrac{1}{4}\sum_L \bar{n}_L L(L-1) \tag{2}$$

and

$$\gamma_G' - 1 = -\tfrac{1}{2}(\mathcal{L} + V_s - 1) - \tfrac{1}{4}\sum_L \bar{n}_L L(L-1) - \tfrac{1}{2}\sum_L \bar{n}'_L L^2 \qquad (3)$$

where V_s is the number of vertices fixed in the surface and $\mathcal{L}$ is the number of loops in G, which is given by Euler's law

$$\mathcal{L} = N - V + 1 = \sum_{L\geqslant 1} \tfrac{1}{2}(L-2)(n_L + n'_L) + 1. \qquad (4)$$

Here $n_L(n'_L)$ is the number of L-leg vertices in the bulk (surface) and V is the total number of vertices.

These results are obtained by establishing a correspondence with the vicious random walker problem for which exact results are known (Fisher 1984, Huse and Fisher 1984). To this end we suppose that the network is embedded in the fully directed square lattice (figure 1). Further let the vertices be partitioned into levels such that the x-coordinate of all vertices in level k is kt ($k = 0, \ldots, n$) and let N_k be the number of non-intersecting chains connecting levels $k-1$ and k, ($\sum_k N_k = N$). For fixed values of the y-coordinates of their end points the number of configurations of these chains may be enumerated independently of the chains connecting other levels. Each such configuration corresponds to the t-step space–time trajectories of a set of N_k vicious lock-step random walkers on a one-dimensional lattice who, at each tick of a clock, move one step to the left or one step to the right but shoot each other on arriving at the same site.

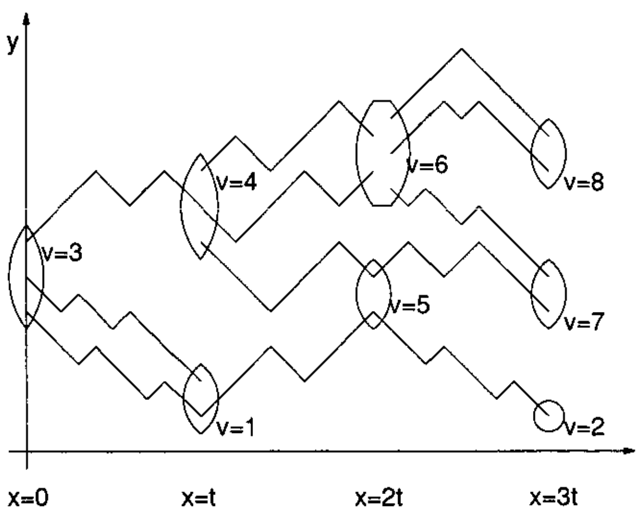

Figure 1. A polymer network embedded in a fully directed square lattice. The vertices are partitioned into levels such that the x-coordinate of vertices in level k is kt, $k = 0, 1, \ldots, n$.

Let the N_k vicious walkers start at $y'_{k-1} = [y'_{k-1,1}, \cdots, y'_{k-1,N_k}]$ and terminate at $y_k = [y_{k,1}, y_{k,2}, \ldots, y_{k,N_k}]$. For sufficiently large t, the number of configurations is approximated by (Huse and Fisher 1984) as

$$
\begin{aligned}
W_{N_k}&(y'_{k-1} \to y_k, t) \\
&= 2^{N_k t} \frac{\exp(-(|y_k|^2 + |y'_{k-1}|^2)/2t)}{(2\pi t)^{N_k/2}} \det(\exp(y'_{k-1,i} y_{k,j}/t))_{i,j=1}^{N_k} \\
&= 2^{N_k t} \frac{\exp(-(|Y_k|^2 + |Y'_{k-1}|^2)/2)}{(2\pi t)^{N_k/2}} \\
&\quad \times \prod_{1 \leqslant i < j \leqslant N_k} (Y'_{k-1,j} - Y'_{k-1,i})(Y_{k,j} - Y_{k,i})(1 + O(t^{-1})) \qquad (5)
\end{aligned}
$$

where Y is the scaled y displacement

$$
Y = yt^{-1/2}. \qquad (6)
$$

To obtain the total number of configurations of G with one vertex fixed we must sum over all values of the y-variables such that $y_{k,i} < y_{k,i+1}$ and which are consistent with the network constraints. Thus there is only one independent variable for all chains which start or end on the same vertex v and we shall take this to be its centre of mass $\bar{y}_v$

We suppose that the ends of chains which belong to the same vertex are symmetrically placed relative to the mass centre and adjacent ends are distance 2 apart (see figure 1). In the case that the number of chains entering or leaving a vertex is even, the ends are shifted by one time step so that the walks are all on the same sublattice. The components of the vectors $\boldsymbol{Y}_k$ and $\boldsymbol{Y}'_{k-1}$ may be replaced by the centre of mass coordinates of the vertices to which they belong, the errors introduced being $O(t^{-1})$ except in the terms of the product where the difference is between two coordinates belonging to the same vertex. In the latter case the centre of mass coordinate cancels and the difference arises from the deviations only.

$$
\begin{aligned}
W(N,t) &= \sum_{\bar{y}_2} \cdots \sum_{\bar{y}_V} \prod_{k=1}^{n} W_{N_k}(y'_{k-1} \to y_k, t) \\
&= \frac{2^{Nt} t^{(V-1)/2}}{(2\pi t)^{N/2}} U(t)(1 + O(t^{-1}) \\
&\quad \int \mathrm{d}\bar{Y}_2 \ldots \int \mathrm{d}\bar{Y}_V \, P(\bar{Y}_1, \ldots, \bar{Y}_V) \prod_{v=1}^{V} \exp(-L_v |\bar{Y}_v|^2/2) \qquad (7)
\end{aligned}
$$

where the repeated sums and integrals are such that $\bar{y}_1 < \bar{y}_2 < \ldots < \bar{y}_V$. The function $P(\bar{Y}_1, \ldots, \bar{Y}_V)$ in the integrand of (7) is the product of all factors in the product of (5) in which the $\boldsymbol{Y}$s belong to different vertices and in which the $\boldsymbol{Y}$s are replaced by the centre of mass coordinate of the vertex to which they belong. The integral contributes only a G-dependent amplitude factor and the local contributions to the critical exponent come from the factor $U(t)$ which is defined as

$$
U(t) = \prod_{v=1}^{V} \left\{ \prod_{1 \leqslant \ell' < m' \leqslant L_v^-} (2(m' - \ell')t^{-1/2}) \prod_{1 \leqslant \ell < m \leqslant L_v^+} (2(m - \ell)t^{-1/2}) \right\} \qquad (8)
$$

where L_v^- is the number of legs in the incoming fan at vertex v and L_v^+ is the number of legs in the outgoing fan. It follows that the critical exponent has the value

$$\gamma_G - 1 = -\frac{1}{2}(N - V + 1) - \frac{1}{4}\sum_{v=1}^{V}[L_v^-(L_v^- - 1) + L_v^+(L_v^+ - 1)] \quad (9)$$

which completes the proof of (2). □

For a *p-star* polymer, $N - V + 1 = 0$, $L_1^+ = p$, $L_i^- = 1$ for $i = 2, \dots, p+1$ and hence $\gamma_G - 1 = -1/4p(p-1)$ in agreement with Fisher (1984, (equation 4.2)). Similarly for a *p-watermelon*, $N - V + 1 = p - 1$, $L_1^+ = L_2^- = p$ and $\gamma_G - 1 = -1/2(p^2 - 1)$ (cf Fisher 1984, (equation 4.3)).

Similarly, the configurations of a network G in which V_s of the vertices are fixed near the surface $y = 0$, can be identified with the trajectories of vicious random walkers in the presence of an absorbing wall. The number of such trajectories and hence polymer configurations becomes (Forrester 1989), instead of (5)

$$\begin{aligned} W_{N_k}^{\mathrm{surf}}&(y'_{k-1} \to y_k, t) \\ &= 2^{N_k t}\frac{\exp(-(|y_k|^2 + |y'_{k-1}|^2)/2t)}{(2\pi t)^{N_k/2}}\det(\sinh(y'_{k-1,i}y_{k,j}/t))_{i,j=1}^{N_k} \\ &= 2^{N_k t}\frac{\exp(-(|Y_k|^2 + |Y'_{k-1}|^2))}{(2\pi t)^{N_k/2}} \\ &\quad \times \prod_{i=1}^{N_k}(Y'_{k-1,i}Y_{k,i}) \prod_{1\leqslant i<j\leqslant N_k}({Y'_{k-1,j}}^2 - {Y'_{k-1,i}}^2)(Y_{k,j}^2 - Y_{k,i}^2) \\ &\quad \times (1 + O(t^{-1})). \end{aligned} \quad (10)$$

We suppose that the endpoints of the chains entering and leaving a vertex v attached to the surface have y-coordinates $2\ell' + \delta'$ and $2\ell + \delta$ respectively where $\ell' = 1, \dots, L_v^-$ and $\ell = 1, \dots, L_v^+$. Also if $L_v^- \geqslant L_v^+$ then $\delta' = 0$ and $\delta = L_v^- - L_v^+$ or if $L_v^+ \geqslant L_v^-$ then $\delta' = L_v^+ - L_v^-$ and $\delta = 0$. By following the same arguments as above, equation (7) is replaced by

$$\begin{aligned} W^{\mathrm{surf}}(N, t) &= \sum_{\bar{y}_{V_s+1}} \cdots \sum_{\bar{y}_V} \prod_{k=1}^{n} W_{N_k}^{\mathrm{surf}}(y'_{k-1} \to y_k, t) \\ &= \frac{2^{Nt} t^{(V-V_s)/2}}{(2\pi t)^{N/2}} U_s(t)(1 + O(t^{-1})) \\ &\quad \times \int \mathrm{d}\bar{Y}_{V_s+1} \cdots \int \mathrm{d}\bar{Y}_V \, P_s(\bar{Y}_1, \dots, \bar{Y}_V) \prod_{v=1}^{V} \exp(-L_v|\bar{Y}_v|^2/2) \end{aligned} \quad (11)$$

where

$$\begin{aligned} U_s(t) = U(t) \prod_{v=1}^{V_s} \Bigg(&\prod_{\ell'=1}^{L_v^-}((2\ell' + \delta')t^{-1/2}) \\ &\times \prod_{1\leqslant \ell'<m'\leqslant L_v^-} (4(m'^2 - \ell'^2)t^{-1}) \prod_{\ell=1}^{L_v^+}((2\ell + \delta)t^{-1/2}) \\ &\times \prod_{1\leqslant \ell<m\leqslant L_v^+} (4(m^2 - \ell^2)t^{-1}) \Bigg) \end{aligned} \quad (12)$$

where $U(t)$ is defined as above but the product is restricted to vertices in the bulk. The critical exponent is therefore

$$\gamma_G' - 1 = -\frac{1}{2}(N - V + V_s) - \frac{1}{4}\sum_{v=V_s+1}^{V}[L_v^-(L_v^- - 1) + L_v^+(L_v^+ - 1)] - \frac{1}{2}\sum_{v=1}^{V_s}[(L_v^-)^2 + (L_v^+)^2] \qquad (13)$$

which agrees with (3). For the *p-star* polymer in which the vertex of degree p is embedded in the surface, $\gamma_G - 1 = -1/2p^2$ (cf Forrester 1989 (equation 31)) and for the *p-watermelon* $\gamma_G - 1 = -(3p^2 + p - 2)/4$ (cf Forrester 1989 (equation 28)).

One of us, JWE, wishes to thank the Centre for Chemical Physics at the University of Western Ontario for support and hospitality during this research. This work was partially supported by NSERC of Canada.

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